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NONLINEAR BERNSTEIN-GREENE-KRUSKAL WAVE EQUILIBRIA
SUBJECT TO GLOBAL ENERGY
AND MOMENTUM CONSERVATION CONSTRAINTS

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The class of Bernstein-Greene-Kruskal (BGK) solutions to the nonlinear Vlasov-Poisson equations are examined within the context of the conservation of (spatially averaged) number, momentum and total

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20. Abstract, continued

energy, imposed as ancillary global constraints that connect the final saturated BGK state to a specified initial distribution function $f(x,v,0)$. While imposing three conservation constraints of course does not uniquely determine the final BGK state, it does remove a large degree of ambiguity as to whether particular classes of solutions are accessible from given initial conditions. It also permits a determination of important features of the final BGK state (e.g., saturation amplitude, wave phase velocity, etc.) in terms of properties of the initial distribution function. The specific example of a low-density electron beam propagating through a background plasma with $f(x,v,0) = (n_0-n_b)\delta(v) + n_b\delta(v-V_b)$ is analyzed in detail assuming a saturated BGK state with monochromatic waveform $\phi_0(x') = \hat{\phi}_0 \cos kx'$, and monoenergetic distribution of untrapped electrons with $F_u^{\pm}(H) = \hat{n}_u(2mH_0)^{1/2}\delta(H-H_0)$ in the wave frame. Here $x' = x - (\omega/k)t$ and $H = (m/2)(v-\omega/k)^2 - e\phi_0(x')$ is the energy.

1. INTRODUCTION AND SUMMARY

The class of Bernstein-Greene-Kruskal (BGK) solutions[1] to the Vlasov-Poisson equations refer to exact nonlinear traveling-wave solutions that have a stationary waveform in a frame of reference moving with constant phase velocity $v_p = \omega/k = \text{const}$. The existence of such solutions has been known for some time,[1-4] and their properties analyzed theoretically in various regimes of interest. Nonetheless, nonlinear BGK waves have remained primarily a curiosity in the annals of plasma physics, particularly because of the great generality and latitude in specifying the final BGK state, and the lack of connection to initial conditions and system dynamics prior to saturation. The purpose of this article is to reexamine the class of BGK solutions to the nonlinear Vlasov-Poisson equations within the context of the conservation of (spatially averaged) number, momentum and total energy, imposed as ancillary global constraints that connect the final saturated BGK state to a specified initial distribution function $f(x,v,0)$. While imposing three conservation constraints of course does not uniquely determine the final BGK state, it does remove a large degree of ambiguity as to whether particular classes of solutions are accessible from given initial conditions. It also permits a determination of important features of the final BGK state (e.g., saturation amplitude, wave phase velocity, etc.) in terms of properties of the initial distribution function.

The organization of this paper is the following. In Secs. 2 and 3, we review the general class of BGK solutions to the nonlinear Vlasov-Poisson equations, including the conservation of average number [Eq. (3)], momentum [Eqs. (4) and (25)], and total energy [Eqs. (5) and (26)] as additional global constraint conditions. For simplicity,

the positive ions are treated as an infinitely massive ($m_i \to \infty$) background that provides overall charge neutrality. In Sec. 3, for specificity, it is assumed that the potential waveform of the saturated state is purely monochromatic with [Eq. (10)]

$$\phi_0(x') = \hat{\phi}_0 \cos kx',$$

where $x' = x - v_p t$, $v_p = \omega/k =$ const. is the phase velocity, and $\hat{\phi}_0 =$ const. is the wave amplitude. In this regard, it should be emphasized that completely analogous constraint conditions exist for general waveform $\phi_0(x')$. The specific choice $\phi_0(x') = \hat{\phi}_0 \cos kx'$ is used to illustrate the method. To summarize the essential results in Sec. 3, we note the following. First, from Eqs. (23) and (24), a specification of the distribution of untrapped electrons $F_u^+(H) = f_>(H) + f_<(H)$ [for $e\hat{\phi}_0 < H < \infty$] in the saturated state uniquely determines the distribution of trapped electrons $F_T(H)$ [for $-e\phi_0(x') < H < e\hat{\phi}_0$]. Here, $H = (m/2)v'^2 - e\phi_0(x')$ is the energy in the wave frame. Second, for the periodic waveform $\phi_0(x') = \hat{\phi}_0 \cos kx'$, number conservation follows trivially from Poisson's equation with $C_1 = n_0$ [Eq. (21)]. Finally, the untrapped- and trapped-particle distribution functions $F_u^{\pm}(H)$ and $F_T(H)$ are related to the initial conditions $f(x,v,0)$ through momentum and energy conservation [Eqs. (25) and (26)]. It is these additional constraint equations which reduce the generality of the final saturated BGK state and connect properties of the saturated state to the initial conditions.

To illustrate with a specific example, in Secs. 4 and 5 we consider initial conditions corresponding to a low-density electron beam propagating through a cold background plasma. For small initial

perturbations, the initial electron distribution function is given by [Eq. (31)]

$$f(x,v,0) = (n_0-n_b)\delta(v) + n_b\delta(v-V_b),$$

in the laboratory frame. It is also assumed that the saturated state is characterized by monoenergetic, backward-moving ($v' = v-v_p < 0$) electrons in the wave frame with untrapped distribution function [Eq. (33)]

$$F_u^{\pm}(H) = \hat{n}_0(2mH_0)^{1/2}\delta(H-H_0),$$

for $e\hat{\phi}0 < H < \infty$. Here n_0, n_b, V_b, $\hat{n}_u$, and H_0 are constants. Making use of Eqs. (22) and (33), the untrapped electron density $n_u(x')$, trapped electron density $n_T(x')$, and trapped electron distribution function $F_T(H)$ are exactly determined by Eqs. (34) - (37). As a final assumption (unrelated to the conservation constraints), we consider the case where the entire beam is trapped by the electrostatic potential, i.e., $\langle n_T(x')\rangle = n_b$, where $\langle\ldots\rangle = \int_0^{2\pi/k} \frac{d(kx')}{2\pi}\ldots$ This relates the constant $\hat{n}_u$ in Eq. (33) to n_0, n_b, and the dimensionless parameter $e\hat{\phi}_0/H_0$ [Eq. (41)].

In Secs. 4 and 5, we also make use of Eqs. (31), (36), and (41), and the expressions for $F_T(H)$ [Eq. (37)] and $F_u^{\pm}(H)$ [Eq. (33)], to simplify the momentum and energy constraints in Eqs. (25) and (26). This leads to three equations [Eqs. (42), (48), and (49)] which connect the four quantities $\hat{\phi}_0$ (saturation amplitude), H_0 (untrapped electron energy in the wave frame), $v_p = \omega/k$ (wave phase velocity) and $k^2\phi_0^2/16\pi$ (average electric field energy density) to the initial beam kinetic energy density $n_b m V_b^2/2$, beam density n_b, and total plasma density n_0. These three equations with four unknowns permit a unique determination of the three dimensionless quantities $e\hat{\phi}_0/H_0$, $k^2\hat{\phi}_0^2/16\pi n_0 H_0 \equiv \hat{W}_F$,

and $v_p/V_0 = (mv_p^2/2H_0)^{1/2}$ characterizing the final state in terms of normalized beam density n_b/n_0 and normalized beam kinetic energy $n_b mV_b^2/2n_0H_0 \equiv \hat{K}_b$ [Eqs. (42), (53), and (54)]. For example, shown in Figs. 3 and 4, are plots of the $\hat{W}_F$ = const. and $\hat{K}_b$ = const. universal contours in the parameter space $(e\phi_0/H_0, n_b/n_0)$ obtained numerically from Eqs. (42) and (53),

$$\hat{W}_F \equiv \frac{k^2\hat{\phi}_0^2}{16\pi n_0H_0} = \frac{1}{4}\frac{e\hat{\phi}_0}{H_0}\left[\left(1-\frac{n_b}{n_0}\right)\frac{\pi(1+e\hat{\phi}_0/H_0)^{1/2}}{F(\pi,\kappa)(1-e\hat{\phi}_0/H_0)^{1/2}}-1\right] ,$$

and

$$\hat{K}_b \equiv \frac{n_b mV_b^2/2}{n_0H_0} = \frac{\pi(1+e\hat{\phi}_0/H_0)^{1/2}}{F(\pi,\kappa)(1-e\hat{\phi}_0/H_0)^{1/2}}\left(1-\frac{5}{8}\frac{e\hat{\phi}_0}{H_0}\right) + \frac{1}{8}\frac{e\hat{\phi}_0/H_0}{(1-n_b/n_0)} - \left(1-\frac{n_b}{n_0}\right)\frac{\pi^2(1+e\hat{\phi}_0/H_0)}{F^2(\pi,\kappa)} ,$$

where $F(\pi,\kappa)$ is the elliptic integral defined in Eq. (40) with $\kappa^2 = (2e\hat{\phi}_0/H_0)/(1+e\hat{\phi}_0/H_0)$. If values of the dimensionless beam parameters n_b/n_0 and $\hat{K}_b$ are prescribed, then the corresponding self-consistent values of $e\hat{\phi}_0/H_0$ and $\hat{W}_F$ follow directly from Fig. 4 [Eq. (53)] and Fig. 3 [Eq. (42)], respectively. For small beam density and potential amplitude with n_b/n_0, $e\hat{\phi}_0/H_0 \ll 1$, the exact expressions for $\hat{W}_F$ [Eq. (42)] and $\hat{K}_b$ [Eq. (53)] reduce approximately to Eqs. (57) and (58), which can be used to estimate (for example) $e\hat{\phi}_0/H_0$ directly in terms of n_b/n_0 and $n_b mV_b^2/2n_0H_0$ [Eq. (59)].

To summarize, for the specific example analyzed in Secs. 4 and 5, it is clear that the ancillary application of global conservation constraints can provide a very useful connection between the initial plasma state and properties of the final saturated BGK state (such as saturation amplitude, average field energy density, etc.). Of course,

the present theory does not prove that a particular saturated BGK state is dynamically accessible from assumed initial conditions. However, if a particular saturated BGK state is dynamically accessible, the present type of analysis does remove a large degree of the ambiguity associated with the final state and connects the saturation amplitude, wave phase velocity, etc., directly to initial properties of the plasma.

We reiterate that the specific choice of $\phi_0(x')$, $F_u^{\pm}(H)$ and $f(x,v,0)$ were used to illustrate the practical utility of the method, and other examples can be analyzed in a completely analogous manner.

2. THEORETICAL MODEL AND ASSUMPTIONS

The present analysis assumes electrostatic perturbations in a nonrelativistic unmagnetized plasma with one-dimensional spatial variations ($\partial/\partial x \neq 0$). The ions are treated as an infinitely massive ($m_i \to \infty$) background with uniform density n_0 that provides overall charge neutrality. In the two-dimensional phase space (x,v), the electron distribution function $f(x,v,t)$ evolves according to the Vlasov equation,

$$\left\{\frac{\partial}{\partial t} + v\frac{\partial}{\partial x} + \frac{e}{m}\frac{\partial\phi}{\partial x}\frac{\partial}{\partial v}\right\} f(x,v,t) = 0 \ , \tag{1}$$

where $E(x,t) = -\partial\phi(x,t)/\partial x$ is the electric field, and $\phi(x,t)$ solves Poisson's equation,

$$\frac{\partial^2\phi}{\partial x^2} = 4\pi e\left[\int_{-\infty}^{\infty} dv f(x,v,t) - n_0\right] \ . \tag{2}$$

In Eqs. (1) and (2), $-e$ is the electron charge and m is the electron rest mass.

Equations (1) and (2) possess three exact nonlinear global invariants corresponding to conservation of particle number, momentum, and total energy. Assuming that $f(x,v,t)$ and $\phi(x,t)$ are characterized by some basic spatial periodicity length L, these conservation relations can be expressed as

$$\int_L \frac{dx}{L} \int_{-\infty}^{\infty} dv\, f\,(x,v,t) = C_1 = \text{const.}, \tag{3}$$

$$\int_L \frac{dx}{L} \int_{-\infty}^{\infty} dv\, v\, f(x,v,t) = C_2 = \text{const.}, \tag{4}$$

$$\int_L \frac{dx}{L} \left(\int_{-\infty}^{\infty} dv\, \frac{m}{2} v^2 f\,(x,v,t) + \frac{1}{8\pi} \left(\frac{\partial}{\partial x} \phi(x,t) \right)^2 \right) = C_3 = \text{const.}, \tag{5}$$

where $C_1 = n_0$ follows from overall charge neutrality [Eq. (2)], and $\int_L dx$ extends over the basic periodicity length L. However complicated the nonlinear evolution of $f(x,v,t)$ and $\phi(x,t)$, the quantities C_1, C_2, and C_3 defined in Eqs. (3) - (5) remain exactly conserved, and the values of C_1, C_2, and C_3 can be determined directly in terms of the initial conditions $f(x,v,0)$ and $\phi(x,0)$, with $C_1 = n_0$ as previously noted.

With regard to the nonlinear BGK analysis of Eqs. (1) and (2) presented in Secs. 3-5, we will find that the conservation equations (3)-(5) provide very important nonlinear constraints on the evolution of the system. Indeed, these constraints greatly reduce the generality and arbitrary nature of the saturated BGK solutions for specified initial conditions.

3. NONLINEAR TRAVELING WAVE SOLUTIONS

A. Solutions to Vlasov-Poisson Equations

In the traditional sense of Bernstein, Greene and Kruskal,[1-3] we investigate saturated traveling-wave solutions to the nonlinear Vlasov-Poisson equations (1) and (2) that are stationary in a frame of reference moving with phase velocity

$$v_p = \frac{\omega}{k} = \text{const.} \tag{6}$$

Defining

$$\begin{aligned} x' &= x - v_p t \ , \\ v' &= v - v_p \ , \\ t' &= t \ , \end{aligned} \tag{7}$$

and denoting the stationary solutions ($\partial/\partial t' = 0$) in the moving frame by $f_0(x',v')$ and $\phi_0(x')$, it is readily shown from Eqs. (1) and (2) that $f_0(x',v')$ and $\phi_0(x')$ satisfy

$$\left\{ v' \frac{\partial}{\partial x'} + \frac{e}{m} \frac{\partial \phi_0(x')}{\partial x'} \frac{\partial}{\partial v'} \right\} f_0(x',v') = 0 \ , \tag{8}$$

$$\frac{\partial^2 \phi_0(x')}{\partial x'^2} = 4\pi e \left(\int_{-\infty}^{\infty} dv' \ f_0(x',v') - n_0 \right) \ , \tag{9}$$

where $n(x') = \int_{-\infty}^{\infty} dv' f_0(x',v')$ is the electron density. For specificity, we restrict the subsequent analysis to the case where $\phi_0(x')$ has the waveform

$$\phi_0(x') = \hat{\phi}_0 \cos kx' \ , \tag{10}$$

where the constants $\hat{\phi}_0$ and k are taken to be positive without loss of generality. That is, it is assumed that the nonlinear saturated state $\phi_0(x')$ has a purely monochromatic waveform with constant amplitude $\hat{\phi}_0$ = const. It should be emphasized, however, that the formalism and procedures developed in the present analysis, including use of the ancillary constraint equations (3) - (5), can also be applied to more general nonlinear periodic functions $\phi_0(x')$.

The general nonlinear solution to Eq. (8) can be expressed as

$$f_0(x'v') = f_>(H)\,\Theta(v') + f_<(H)\,\Theta(-v') \ , \tag{11}$$

where H is the energy,

$$H = \frac{m}{2} v'^2 - e\phi_0(x') \ , \tag{12}$$

and $\Theta(v')$ is the Heaviside step function,

$$\Theta(v') = \begin{cases} +1 \ , & v' > 0 \ , \\ 0 \ , & v' < 0 \ . \end{cases} \tag{13}$$

For the electrons trapped in the electrostatic potential $[-e\phi_0(x') < H < e\hat{\phi}_0]$, it necessarily follows that the phase-space density of forward-moving ($v' > 0$) and backward-moving ($v' < 0$) particles are equal. That is, $f_>(H) = f_<(H)$ for the trapped electrons, and the phase-space density of <u>trapped</u> electrons can be expressed as

$$f_T(x',v') = \frac{1}{2} F_T(H)[\Theta(v') + \Theta(-v')] \ ,$$
$$\text{for } -e\phi_0(x') < H < e\hat{\phi}_0 \ , \tag{14}$$

where

$$F_T(H) \equiv 2f_>(H) = 2f_<(H) \;, \tag{15}$$

for $-e\phi_0(x') < H < e\hat{\phi}_0$. On the other hand, for the untrapped electrons $[e\hat{\phi}_0 < H < \infty]$, $f_>(H)$ and $f_<(H)$ can be specified independently, and the phase-space density of <u>untrapped</u> electrons can be expressed as

$$\begin{aligned} f_u(x',v') &= \frac{1}{2} F_u^+(H)[\Theta(v') + \Theta(-v')] \\ &\quad + \frac{1}{2} F_u^-(H)[\Theta(v') - \Theta(-v')] \;, \end{aligned} \tag{16}$$

$$\text{for } e\hat{\phi}_0 < H < \infty,$$

where

$$\begin{aligned} F_u^+(H) &\equiv f_>(H) + f_<(H) \;, \\ F_u^-(H) &\equiv f_>(H) - f_<(H) \;, \end{aligned} \tag{17}$$

for $e\hat{\phi}_0 < H < \infty$.

Making use of Eqs. (14) - (17) and $dv' = \pm dH'/[2m(H + e\phi_0(x'))]^{1/2}$, it is readily shown from Eqs. (9) - (17) that the Poisson equation (9) can be expressed as

$$-k^2\phi_0(x') = 4\pi e[n_u(x') + n_T(x') - n_0] \;, \tag{18}$$

where the untrapped- and trapped-electron densities are defined by

$$n_u(x') = \int_{e\hat{\phi}_0}^{\infty} \frac{dH\; F_u^+(H)}{[2m(H + e\phi_0(x'))]^{1/2}} \;, \tag{19}$$

and

$$n_T(x') = \int_{-e\phi_0(x')}^{e\hat{\phi}_0} \frac{dH\; F_T(H)}{[2m(H + e\phi_0(x'))]^{1/2}} \;. \tag{20}$$

In Eqs. (18) - (20), the electrostatic potential is $\phi_0(x') = \hat{\phi}_0 \cos kx'$, and the integral constraint

$$\int_L \frac{dx'}{L} [n_u(x') + n_T(x')] = n_0 \equiv C_1 \;, \tag{21}$$

follows directly from Eqs. (3), (9), and (10), where $L = 2\pi/k$ and $\int_L dx' = \int_0^{2\pi/k} dx'$.

Note from Eq. (18) that once $n_u(x')$ is calculated from Eq. (19), then the density of trapped electrons is determined directly from Eq. (18) with

$$n_T(x') = n_0 - n_u(x') - \frac{k^2}{4\pi e}\,\hat{\phi}_0 \cos kx' . \tag{22}$$

Moreover, a particularly important theorem can be proved from Eqs. (18) - (20). For specified untrapped distribution $F_u^+(H) = f_>(H) + f_<(H)[e\hat{\phi}_0 < H < \infty]$, it is straightforward to show that the distribution of trapped electrons is uniquely determined from[2]

$$F_T(H) = \frac{(2m)^{1/2}}{\pi}\int_{e\hat{\phi}_0}^{H} dH' \,\frac{1}{[H'-H]^{1/2}}\left(\frac{k^2}{4\pi e^2} - \frac{1}{2}\int_{e\hat{\phi}_0}^{\infty} \frac{dH'' F_u^+(H'')}{[2m(H''-H)]^{1/2}}\right) , \tag{23}$$

$$\text{for } -e\phi_0(x') < H < e\hat{\phi}_0 ,$$

subject to the boundary condition[2]

$$\begin{aligned} 0 = g(-e\hat{\phi}_0) &= n_T[kx' = (2n+1)\pi] \\ &= n_0 + \frac{k^2\hat{\phi}_0}{4\pi e} - \int_{e\hat{\phi}_0}^{\infty} \frac{dH\, F_u^+(H)}{[2m(H-e\hat{\phi}_0)]^{1/2}} , \end{aligned} \tag{24}$$

where $n = 0, 1, 2, \ldots$, is an integer.

To summarize, for specified untrapped distribution $F_u^+(H) = f_>(H) + f_<(H)\ [e\hat{\phi}_0 < H < \infty]$, and monochromatic waveform $\phi_0(x') = \hat{\phi}_0 \cos kx'$, the trapped-electron distribution function $F_T(H)$ is determined from Eq. (23), subject to the constraint on wavenumber k, amplitude $\hat{\phi}_0$, and untrapped distribution $F_u^+(H)$ imposed by Eq. (24).

B. Constraint Equations

Assuming the saturated BGK equilibrium specified by Eqs. (10), (14), (16), and (23) is accessible from given initial conditions $f(x,v,0)$ and $\phi(x,0)$, the conservation relations (3) - (5) provide important ancillary constraints on the nonlinear evolution of the system. The density conservation constraint (21) with $C_1 = n_0$ follows trivially from Eqs. (18) - (20) and (22) for the periodic potential $\phi_0(x') = \hat{\phi}_0 \cos kx'$. Expressed in a frame of reference moving with velocity $v_p = \omega/k$, the momentum and energy constraints (4) and (5) can be expressed after some straightforward algebra as

$$\int_L \frac{dx'}{L} \frac{1}{m} \int_{e\hat{\phi}_0}^{\infty} dH\, F_u^-(H) = C_2 - C_1 v_p \ , \tag{25}$$

and

$$\int_L \frac{dx'}{L}\left[\int_{-e\phi_0}^{e\hat{\phi}_0} \frac{dH\, H\, F_T(H)}{[2m(H+e\phi_0)]^{1/2}} + \int_{e\hat{\phi}_0}^{\infty} \frac{dH\, H\, F_u^+(H)}{[2m(H+e\phi_0)]}\right] - \frac{1}{2}\frac{k^2\hat{\phi}_0^2}{8\pi}$$
$$= C_3 - C_1 \frac{m}{2} v_p^2 - mv_p(C_2 - C_1 v_p) \ , \tag{26}$$

where $\phi_0 = \hat{\phi}_0 \cos kx'$, $\int_L \frac{dx'}{L} = \int_0^{2\pi/k} \frac{d(kx')}{2\pi}$, and $C_1 = n_0$. In obtaining Eqs. (25) and (26), use has been made of the identities $v = v' + v_p$, $mv^2/2 = mv'^2/2 + mv_p^2/2 + mv'v_p = H + mv_p^2/2 + mv'v_p + e\phi_0(x')$, $\int_L \frac{dx'}{L} \phi_0^2(x') = \hat{\phi}_0^2/2$, and $\int_L \frac{dx'}{L} \phi_0(x')[n_T(x') + n_u(x')] = -(k^2/4\pi e)\int_L \frac{dx'}{L} \phi_0^2(x') = -k^2\hat{\phi}_0^2/8\pi e$ [Eq. (18)]. Moreover, $F_T(H)$ and $F_u^{\pm}(H)$ are defined in Eqs. (15) and (17).

To simplify the subsequent analytic calculations, it is useful to define

$$H' = H + e\phi_0(x') \ , \tag{27}$$

and

$$\Phi(x') = \phi_0(x') + \hat{\phi}_0 \ . \tag{28}$$

In this case, the conservation of momentum and energy, Eqs. (25) and (26), can be expressed in the equivalent form

$$\int_L \frac{dx'}{L} \frac{1}{m} \int_{e\Phi}^{\infty} dH' F_u^-(H'-e\phi_0) = C_2 - n_0 v_p \,, \tag{29}$$

and

$$\int_L \frac{dx'}{L} \left(\int_0^{e\Phi} \frac{dH' H' F_T(H'-e\phi_0)}{(2mH')^{1/2}} + \int_{e\Phi}^{\infty} \frac{dH' H' F_u^+(H'-e\phi_0)}{(2mH')^{1/2}} \right) + \frac{1}{2} \frac{k^2 \hat{\phi}_0^2}{8\pi}$$
$$= C_3 - n_0 \frac{m}{2} v_p^2 - m v_p (C_2 - n_0 v_p). \tag{30}$$

To summarize, a specification of initial conditions $f(x,v,0)$ and $\phi(x,0)$ determines the values of C_2 and C_3 from Eqs. (4) and (5). Moreover, for given untrapped distribution function $F_u^{\pm}(H)$ and monochromatic waveform $\phi_0(x') = \hat{\phi}_0 \cos kx'$, the trapped-particle distribution function $F_T(H)$ is uniquely determined from Eqs. (23) and (24). Finally, the distributions $F_T(H)$ and $F_u^{\pm}(H)$ in the saturated state are further connected to the initial conditions through the momentum and energy constraint equations (29) and (30). As will become evident in Sec. 4, these constraint equations greatly reduce the generality and arbitrariness of the final saturated state for given initial conditions.

4. ANALYTIC EXAMPLE OF SATURATED BGK EQUILIBRIUM

A. Initial Conditions and Saturated State

It is clear that the formalism developed in Secs. 2 and 3 can be applied to a wide variety of initial conditions $f(x,v,0)$ and saturated states $f_0(x',v')$ subject to the constraint equations (29) and (30) and the assumption of monochromatic waveform $\phi_0(x') = \hat{\phi}_0 \cos kx'$ [Eq. (10)]. For purposes of illustration, we consider the example of a low-density ($n_b < n_0$) electron beam propagating through a cold background plasma. For small initial perturbations, the initial electron distribution function is modeled by [Fig. 1],

$$f(x,v,t=0) = (n_0-n_b)\delta(v) + n_b\delta(v-V_b) \ , \tag{31}$$

in the laboratory frame. From Eqs. (4), (5), and (31), the constants C_2 and C_3 are given by

$$\begin{aligned} C_2 &= n_b V_b \ , \\ C_3 &= n_b \frac{m}{2} V_b^2 \ , \end{aligned} \tag{32}$$

for the example considered here.

For present purposes, it is also assumed that the saturated state is characterized by monoenergetic backward-moving ($v' < 0$) electrons in the wave frame with $f_>(H) = 0$ and $f_<(H) = \hat{n}_u (2mH_0)^{1/2}\delta(H-H_0)$, for $e\hat{\phi}_0 < H < \infty$. Here, $\hat{n}_u$ and $H_0 = mV_0^2/2$ are positive constants, and from Eq. (17) the untrapped distribution function $F_u^{\pm}(H)$ can be expressed as

$$F_u^+(H) = F_u^-(H) = \hat{n}_u (2mH_0)^{1/2}\delta(H-H_0), \quad \text{for } e\hat{\phi}_0 < H < \infty. \tag{33}$$

Making use of Eqs. (19) and (33), the density of untrapped electrons is given by

$$n_u(x') = \frac{\hat{n}_u}{[1+e\phi_0(x')/H_0]^{1/2}} , \tag{34}$$

and from Eqs. (22) and (34) the density of trapped electrons is given by

$$n_T(x') = n_0 - \frac{k^2}{4\pi e}\hat{\phi}_0\cos kx' - \frac{\hat{n}_u}{[1+e\phi_0(x')/H_0]^{1/2}} , \tag{35}$$

where $\phi_0(x') = \hat{\phi}_0\cos kx'$. For the choice of untrapped distribution function $F_u^+(H)$ in Eq. (33), the constraint condition (24) can be expressed as

$$\frac{k^2\hat{\phi}_0}{4\pi e} = \frac{\hat{n}_u}{(1-e\hat{\phi}_0/H_0)^{1/2}} - n_0 , \tag{36}$$

which is equivalent to $n_T[kx'=(2n+1)\pi] = 0$ for $n = 0, \pm 1, \pm 2, \ldots$ [Eq. (35)].

Finally, for the choice of $F_u^{\pm}(H)$ in Eq. (33), it can be shown after some straightforward algebra that the corresponding self-consistent distribution of trapped electrons $F_T(H)$ is given by [Eq. (23)]

$$F_T(H) = \frac{(2m)^{1/2}}{\pi}(e\hat{\phi}_0-H)^{1/2}\left(-\frac{k^2}{2\pi e^2} + \frac{\hat{n}_u}{(H_0-H)}\frac{1}{(1-e\hat{\phi}_0/H_0)^{1/2}}\right) , \tag{37}$$

$$\text{for } -e\phi_0(x') < H < e\hat{\phi}_0 .$$

Note from Eq. (37) that $F_T(H=e\hat{\phi}_0) = 0$ [Fig. 2]. Moreover, the quantity k^2 occurring in Eq. (37) can be eliminated in favor of $\hat{\phi}_0$, $\hat{n}_u$, n_0, etc., by making use of Eq. (36).

From Eq. (35), the average density of trapped electrons $\langle n_T(x')\rangle = \int_L \frac{dx'}{L} n_T(x')$ can be expressed as

$$\begin{aligned}\langle n_T(x')\rangle &= n_0 - \langle n_u(x')\rangle \\ &= n_0 - \hat{n}_u \frac{1}{\pi} \frac{F(\pi,\kappa)}{(1+e\hat{\phi}_0/H_0)^{1/2}} ,\end{aligned} \tag{38}$$

where

$$\kappa^2 = \frac{2e\hat{\phi}_0/H_0}{(1+e\hat{\phi}_0/H_0)} , \tag{39}$$

and $F(\pi,\kappa)$ is the elliptic integral defined by

$$F(\pi,\kappa) = \int_0^\pi d\alpha \frac{1}{(1-\kappa^2\sin^2\alpha)^{1/2}} . \tag{40}$$

If we impose the additional constraint that the entire electron beam is trapped by the electrostatic wave, i.e., $\langle n_T(x')\rangle = n_b$, then it follows from Eq. (38) that

$$\hat{n}_u \frac{1}{\pi} \frac{F(\pi,\kappa)}{(1+e\hat{\phi}_0/H_0)^{1/2}} = n_0 - n_b , \tag{41}$$

which expresses $\hat{n}_u$ directly in terms of n_0, n_b and $e\hat{\phi}_0/H_0$.

For present purposes, we assume that Eq. (41) is satisfied, which is equivalent to the requirement that the entire beam is trapped by the electrostatic wave. In this regard, it should be emphasized that Eq. (41) is an ancillary constraint imposed in addition to the conservation of momentum and energy in Eqs. (29) and (30). For future reference, it is convenient to eliminate the constant $\hat{n}_u$ from Eqs. (36) and (41). Defining $\hat{W}_F \equiv k^2\hat{\phi}_0^2/16\pi n_0 H_0$, which is a dimensionless

measure of the average electric field energy density, it follows from Eqs. (36) and (41) that

$$\hat{W}_F \equiv \frac{k^2\hat{\phi}_0^2}{16\pi n_0 H_0} = \frac{1}{4}\frac{e\hat{\phi}_0}{H_0}\left[\left(1 - \frac{n_b}{n_0}\right)\frac{\pi(1+e\hat{\phi}_0/H_0)^{1/2}}{F(\pi,\kappa)(1-e\hat{\phi}_0/H_0)^{1/2}} - 1\right] . \tag{42}$$

From Eq. (42), it is straightforward to determine numerically [Fig. 3] the allowed $\hat{W}_F$=const. contours in the parameter space $(e\hat{\phi}_0/H_0,\ n_b/n_0)$. With regard to Fig. 3, the $\hat{W}_F$=const. contours are plotted for parameters spanning the range $0 < e\hat{\phi}_0/H_0 < 1$ and $0 < n_b/n_0 < 1$. (Keep in mind that $\hat{\phi}_0 > 0$ has been assumed without loss of generality.) Moreover, since $\hat{W}_F \geq 0$ is required for physically allowed solutions, it follows from Eq. (42) that

$$\frac{n_b}{n_0} \leq 1 - \frac{F(\pi,\kappa)(1-e\hat{\phi}_0/H_0)^{1/2}}{\pi(1+e\hat{\phi}_0/H_0)^{1/2}} . \tag{43}$$

In Sec. 4.B, we will find that energy and momentum conservation will further restrict the allowed values of $(\hat{W}_F,\ e\hat{\phi}_0/H_0,\ n_b/n_0)$ relative to the constraint in Eq. (42).

B. Energy and Momentum Conservation

In this section, we impose the momentum and energy conservation constraints in Eqs. (29) and (30) for the choice of self-consistent distributions $F_u^{\pm}(H)$ and $F_T(H)$ in Eqs. (37) and (33). Substituting Eqs. (33) and (37) into Eqs. (29) and (30), and making use of $C_2 = n_b V_b$ and $C_3 = n_b m V_b^2/2$ for the choice of initial conditions in Eq. (31), it follows after some straightforward algebraic manipulation that

$$-\left(\frac{2H_0}{m}\right)^{1/2}\hat{n}_u = n_bV_b - n_0v_p \ , \tag{44}$$

$$\left[n_0H_0 - \frac{k^2\hat{\phi}_0^2}{8\pi}\right] + \left[\frac{k^2}{2\pi e^2}\left(\frac{1}{2}H_0e\hat{\phi}_0 + \frac{1}{16}e^2\hat{\phi}_0^2\right) - \frac{1}{2}e\hat{\phi}_0\frac{\hat{n}_u}{(1-e\hat{\phi}_0/H_0)^{1/2}}\right]$$
$$+\frac{1}{2}\frac{k^2\hat{\phi}_0^2}{8\pi} = n_0\frac{m}{2}v_p^2 + n_b\frac{m}{2}V_b^2 - n_bmv_pV_b \ . \tag{45}$$

The first square-bracket term in Eq. (45) represents the contribution from the untrapped electrons in Eq. (30), whereas the second square-bracket term represents the contribution from the trapped electrons.

Defining $V_0 = (2H_0/m)^{1/2}$, and making use of Eq. (36) to eliminate $k^2\hat{\phi}_0$ in the term proportional to $k^2H_0e\hat{\phi}_0$ in Eq. (45), the conservation equations (44) and (45) can be expressed in the equivalent form

$$-\hat{n}_uV_0 = n_bV_b - n_0v_p \ , \tag{46}$$

$$\hat{n}_uH_0\frac{(1-e\hat{\phi}_0/H_0)}{(1-e\hat{\phi}_0/H_0)^{1/2}} - \frac{k^2\hat{\phi}_0^2}{32\pi} = n_0\frac{m}{2}v_p^2 + n_b\frac{m}{2}V_b^2 - n_bmv_pV_b \ . \tag{47}$$

If we further make use of Eq. (41) to eliminate $\hat{n}_u$ in favor of n_0, n_b and $e\hat{\phi}_0/H_0$, the conservation equations (46) and (47) become

$$-(n_0-n_b)V_0\frac{\pi(1+e\hat{\phi}_0/H_0)^{1/2}}{F(\pi,\kappa)} = n_bV_b - n_0v_p \ , \tag{48}$$

$$(n_0-n_b)H_0\frac{(1-e\hat{\phi}_0/2H_0)}{(1-e\hat{\phi}_0/H_0)^{1/2}}\frac{\pi(1+e\hat{\phi}_0/H_0)^{1/2}}{F(\pi,\kappa)} - \frac{k^2\hat{\phi}_0^2}{32\pi}$$
$$= n_0\frac{m}{2}v_p^2 + n_b\frac{m}{2}V_b^2 - n_bmv_pV_b \ . \tag{49}$$

To summarize, Eqs. (42), (48), and (49) constitute the final equations of the present theory. These equations connect the saturation amplitude $\hat{\phi}_0$, average electric field energy density $k^2\hat{\phi}_0^2/16\pi$, untrapped

electron energy H_0 in the wave frame, and wave phase velocity $v_p = \omega/k$ to the beam density n_b, total plasma density n_0, and the initial beam kinetic energy density $n_b m V_b^2/2$.

5. ANALYSIS OF CONSTRAINT EQUATIONS

A. General Analysis

As indicated at the end of Sec. 4, the three constraint equations (42), (48), and (49) relate the four unknown quantities $\hat{\phi}_0$, $k^2\hat{\phi}_0^2/16\pi$, H_0, and v_p associated with the final saturated state to the quantities n_0, n_b and $n_b mV_b^2/2$ that characterize the initial distribution of electrons. Three equations with four unknowns permit a unique determination of the three dimensionless quantities $e\hat{\phi}_0/H_0$, $k^2\hat{\phi}_0^2/16\pi n_0 H_0$ and $v_p/V_0 = (mv_p^2/2H_0)^{1/2}$. Defining the dimensionless parameters

$$\hat{n}_b \equiv \frac{n_b}{n_0}\ , \ \hat{V}_b \equiv \frac{V_b}{V_0}\ , \quad \hat{v}_p \equiv \frac{v_p}{V_0}\ , \tag{50}$$

where $V_0 \equiv (2H_0/m)^{1/2}$, and making use of Eq. (42) to eliminate the $k^2\hat{\phi}_0^2/32\pi$ term in Eq. (49), the momentum and energy conservation equations (48) and (49) can be expressed as

$$\left(1 - \frac{n_b}{n_0}\right)\frac{\pi(1+e\hat{\phi}_0/H_0)^{1/2}}{F(\pi,\kappa)} = \hat{v}_p - \hat{n}_b\hat{V}_b\ , \tag{51}$$

$$\left(1 - \frac{n_b}{n_0}\right)\frac{\pi(1+e\hat{\phi}_0/H_0)^{1/2}}{F(\pi,\kappa)(1-e\hat{\phi}_0/H_0)^{1/2}}\left(1 - \frac{5}{8}\frac{e\hat{\phi}_0}{H_0}\right) + \frac{1}{8}\frac{e\hat{\phi}_0}{H_0}$$
$$= \hat{v}_p^2 - 2\hat{n}_b\hat{v}_p\hat{V}_b + \hat{n}_b\hat{V}_b^2. \tag{52}$$

Squaring Eq. (51) and subtracting from Eq. (52), we eliminate $\hat{v}_p^2 - 2\hat{n}_b\hat{v}_p\hat{V}_b$ and obtain a closed expression for the normalized beam kinetic energy density $\hat{K}_b = \hat{n}_b\hat{V}_b^2$ in terms of $e\hat{\phi}_0/H_0$ and n_b/n_0. This

gives

$$\hat{K}_b \equiv \frac{n_b m V_b^2/2}{n_0 H_0} = \frac{\pi(1+e\hat{\phi}_0/H_0)^{1/2}}{F(\pi,\kappa)(1-e\hat{\phi}_0/H_0)^{1/2}}\left(1 - \frac{5}{8}\frac{e\hat{\phi}_0}{H_0}\right) + \frac{1}{8}\frac{e\hat{\phi}_0/H_0}{(1-n_b/n_0)} - \left(1 - \frac{n_b}{n_0}\right)\frac{\pi^2(1+e\hat{\phi}_0/H_0)}{F^2(\pi,\kappa)} \quad (53)$$

Figure 4 illustrates the $\hat{K}_b$ = const. contours obtained numerically from Eq. (53) in the parameter space $(e\hat{\phi}_0/H_0,\ n_b/n_0)$. The universal curves generated in Fig. 4 have the following practical use. For specified values of n_b/n_0 and $\hat{K}_b = n_b m V_b^2/2n_0H_0$, the corresponding self-consistent value of normalized wave amplitude $e\hat{\phi}_0/H_0$ can be determined directly from Fig. 4. For example, for $\hat{K}_b = 0.1$ and $n_b/n_0 = 0.08$, then $e\hat{\phi}_0/H_0 = 0.25$ follows from Fig. 4 and Eq. (53). Moreover, for consistent values of n_b/n_0, $e\hat{\phi}_0/H_0$ and $\hat{K}_b$, Eq. (42) can then be used to calculate the corresponding value of normalized field energy $\hat{W}_F = k^2\hat{\phi}_0^2/16\pi n_0H_0$. For example, for $(\hat{K}_b,\ n_b/n_0,\ e\hat{\phi}_0/H_0) = (0.1,\ 0.08,\ 0.25)$, then $\hat{W}_F = k^2\hat{\phi}_0^2/16\pi n_0H_0 = 0.03$ follows directly from Eq. (42).

Finally, the normalized phase velocity $\hat{v}_p = v_p/V_0$ can be calculated directly from Eq. (51) by making use of Eq. (53) to eliminate $\hat{V}_b$. We find from Eqs. (51) and (53) that

$$\hat{v}_p \equiv \left(\frac{mv_p^2}{2H_0}\right)^{1/2} = \left(\frac{n_b}{n_0}\right)^{1/2}(\hat{K}_b)^{1/2} + \left(1 - \frac{n_b}{n_0}\right)\frac{\pi(1+e\hat{\phi}_0/H_0)^{1/2}}{F(\pi,\kappa)}\,, \quad (54)$$

where $\hat{K}_b(n_b/n_0,\ e\hat{\phi}_0/H_0)$ is defined in Eq. (53). For consistent values of $e\hat{\phi}_0/H_0$, n_b/n_0 and $\hat{K}_b$ determined from Eq. (53), we then utilize Eq. (54) to determine the corresponding value of $\hat{v}_p$. For example, for $(\hat{K}_b,\ n_b/n_0,\ e\hat{\phi}_0/H_0) = (0.1,\ 0.08,\ 0.25)$, it follows from Eq. (54) that $\hat{v}_p = 0.998$.

B. Small-Amplitude Analysis

In Sec. 5.A, we considered the constraint equations (42), (51), and (52) for general range of parameters $\hat{K}_b = n_b m V_b^2/2n_0H_0$, $e\hat{\phi}_0/H_0$ and n_b/n_0. In this section, we consider the case where the beam density and energy are sufficiently small that it is valid to approximate

$$\frac{n_b}{n_0}\ ,\ \frac{e\hat{\phi}_0}{H_0} << 1\ , \tag{55}$$

in the constraint equations. In this case, correct to quadratic order in $(e\hat{\phi}_0/H_0)$, we approximate

$$\begin{aligned} \frac{\pi}{F(\pi,\kappa)} &= 1 - \frac{1}{2}\left(\frac{e\hat{\phi}_0}{H_0}\right) + \frac{3}{16}\left(\frac{e\hat{\phi}_0}{H_0}\right)^2 , \\ (1+e\hat{\phi}_0/H_0)^{1/2} &= 1 + \frac{1}{2}\ \frac{e\hat{\phi}_0}{H_0} - \frac{1}{8}\left(\frac{e\hat{\phi}_0}{H_0}\right)^2 , \\ (1-e\hat{\phi}_0/H_0)^{-1/2} &= 1 + \frac{1}{2}\left(\frac{e\hat{\phi}_0}{H_0}\right) + \frac{3}{8}\left(\frac{e\hat{\phi}_0}{H_0}\right)^2 , \end{aligned} \tag{56}$$

in Eqs. (42), and (51) - (53). Retaining terms to order $(e\hat{\phi}_0/H_0)^2$ and $\hat{n}_b(e\hat{\phi}_0/H_0)$, we find from Eqs. (42) and (53) (for example) that

$$\hat{W}_F = \frac{k^2\hat{\phi}_0^2}{16\pi n_0H_0} = \frac{1}{4}\left(\frac{e\hat{\phi}_0}{H_0}\right)\left[\frac{1}{2}\left(\frac{e\hat{\phi}_0}{H_0}\right) - \frac{n_b}{n_0}\right] , \tag{57}$$

and

$$\hat{K}_b = \frac{n_b m V_b^2/2}{n_0H_0} = \frac{n_b}{n_0} + \frac{1}{4}\left(\frac{e\hat{\phi}_0}{H_0}\right)\left[\left(\frac{e\phi_0}{H_0}\right) + \frac{1}{2}\ \frac{n_b}{n_0}\right] . \tag{58}$$

If $\hat{K}_b$ and n_b/n_0 are viewed as prescribed quantities, then $e\hat{\phi}_0/H_0$ is

determined from Eq. (58) to be

$$4\left(\frac{e\hat{\phi}_0}{H_0}\right) = [\hat{n}_b^2 + 64(\hat{K}_b - \hat{n}_b)]^{1/2} - \hat{n}_b \ , \qquad (59)$$

where $\hat{n}_b \equiv n_b/n_0$, and $\hat{K}_b \geq \hat{n}_b$ is required from Eq. (58) for $e\hat{\phi}_0/H_0 \geq 0$. The expression for $e\hat{\phi}_0/H_0$ in Eq. (59) can then be substituted into Eq. (57) to determine the normalized field energy $\hat{W}_F$.

6. CONCLUSIONS

In Secs. 2 and 3, we reviewed the general class of BGK solutions to the nonlinear Vlasov-Poisson equations, including the conservation of average number [Eq. (3)], momentum [Eqs. (4) and (25)], and total energy [Eqs. (5) and (26)] as additional global constraint conditions. For specificity, it was assumed that the potential waveform of the saturated state is purely monochromatic with [Eq. (10)]

$$\phi_0(x') = \hat{\phi}_0 \cos kx',$$

where $x' = x - v_p t$, $v_p = \omega/k = \text{const.}$ is the phase velocity, and $\hat{\phi}_0 = \text{const.}$ is the wave amplitude. To summarize the essential results in Sec. 3, we note the following. First, from Eqs. (23) and (24), a specification of the distribution of untrapped electrons $F_u^+(H) = f_>(H) + f_<(H)$ (for $e\hat{\phi}_0 < H < \infty$) in the saturated state uniquely determines the distribution of trapped electrons $F_T(H)$ (for $-e\phi_0(x') < H < e\hat{\phi}_0$). [Here, $H = (m/2)v'^2 - e\phi_0(x')$ is the energy in the wave frame.] Second, for the periodic waveform $\phi_0(x') = \hat{\phi}_0 \cos kx'$, number conservation follows trivially from Poisson's equation with $C_1 = n_0$ [Eq. (21)]. Finally, the untrapped- and trapped-particle distribution functions $F_u^{\pm}(H)$ and $F_T(H)$ are related to the initial conditions $f(x,v,0)$ through momentum and energy conservation [Eqs. (25) and (26)]. It is these additional constraint equations which reduce the generality of the final saturated BGK state and connect properties of the saturated state to the initial conditions.

To illustrate with a specific example, in Secs. 4 and 5 we considered initial conditions corresponding to a low-density electron beam propagating through a cold background plasma [Eq. (31)]. It was also assumed that the saturated state is characterized by monoenergetic,

backward-moving electrons in the wave frame [Eq. (33)]. Making use of Eqs. (22) and (33), the untrapped electron density $n_u(x')$, trapped electron density $n_T(x')$, and trapped electron distribution function $F_T(H)$ were exactly determined by Eqs. (34) - (37). As a final assumption (unrelated to the conservation constraints), we considered the case where the entire beam is trapped by the electrostatic potential, i.e., $\langle n_T(x')\rangle = n_b$, where $\langle\ldots\rangle = \int_0^{2\pi/k} \frac{d(kx')}{2\pi} \ldots$ [Eq. (41)].

In Secs. 4 and 5, we also made use of Eqs. (31), (36), and (41), and the expressions for $F_T(H)$ [Eq. (37)] and $F_u^{\pm}(H)$ [Eq. (33)], to simplify the momentum and energy constraints in Eqs. (25) and (26). This led to three equations Eqs. (42), (48), and (49)] which connect the four quantities $\hat{\phi}_0$ (saturation amplitude), H_0 (untrapped electron energy in the wave frame), $v_p = \omega/k$ (wave phase velocity) and $k^2\hat{\phi}_0^2/16\pi$ (average electric field energy density) to the initial beam kinetic energy density $n_b m V_b^2/2$, beam density n_b, and total plasma density n_0. These three equations with four unknowns permitted a unique determination of the three dimensionless quantities $e\hat{\phi}_0/H_0$, $k^2\hat{\phi}_0^2/16\pi n_0 H_0 \equiv \hat{W}_F$ and $v_p/V_0 = (mv_p^2/2H_0)^{1/2}$ characterizing the final state in terms of normalized beam density n_b/n_0 and normalized beam kinetic energy $n_b m V_b^2/2n_0 H_0 \equiv \hat{K}_b$ [Eqs. (42), (53), and (54)]. If values of the dimensionless beam parameters n_b/n_0 and $\hat{K}_b$ are prescribed, then the corresponding self-consistent values of $e\hat{\phi}_0/H_0$ and $\hat{W}_F$ follow directly from Fig. 4 [Eq. (53)] and Fig. 3 [Eq. (42)], respectively.

For the specific example analyzed in Secs. 4 and 5, it is clear that the ancillary application of global conservation constraints can provide a very useful connection between the initial plasma state and properties of the final saturated BGK state (such as saturation amplitude, average field energy density, etc.). Of course, the present theory

does not prove that a particular saturated BGK state is dynamically accessible from assumed initial conditions. However, if a particular saturated BGK state is dynamically accessible, then the present type of analysis does remove a large degree of the ambiguity associated with the final state and connects the saturation amplitude, wave phase velocity, etc., directly to initial properties of the plasma.

We reiterate that the specific choice of $\phi_0(x')$, $F_u^{\pm}(H)$ and $f(x,v,0)$ were used to illustrate the practical utility of the method, and other examples can be analyzed in a completely analogous manner.

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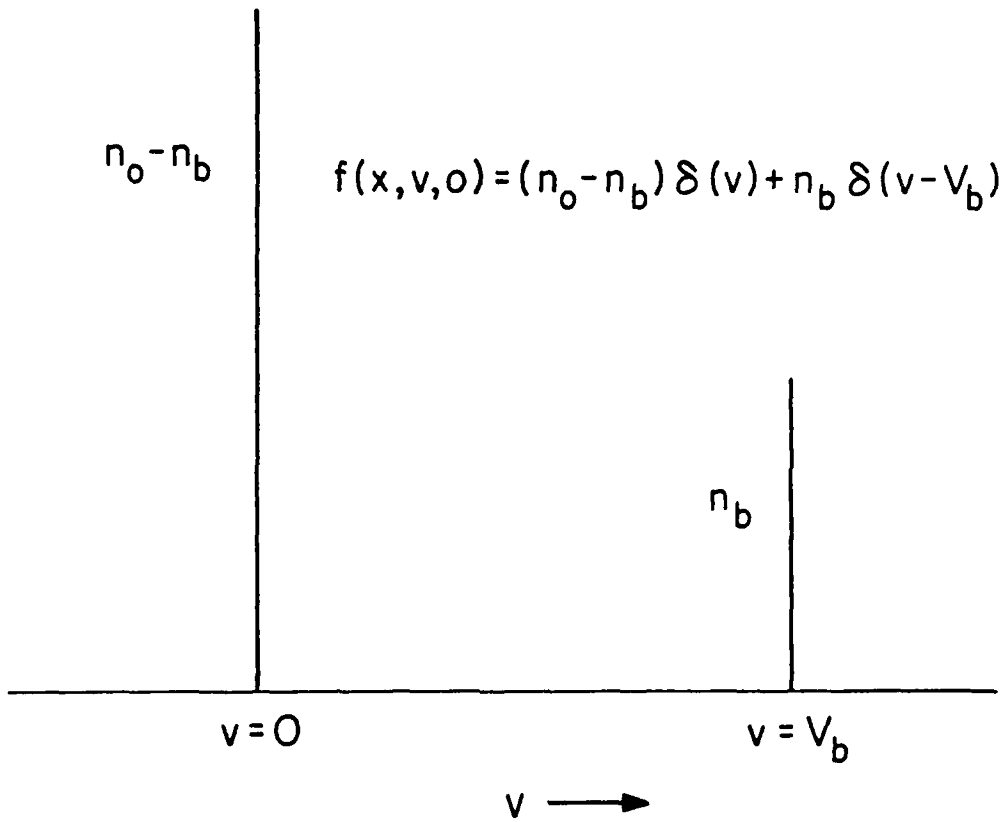

Fig. 1 Initial beam distribution function in the laboratory frame [Eq. (31)].

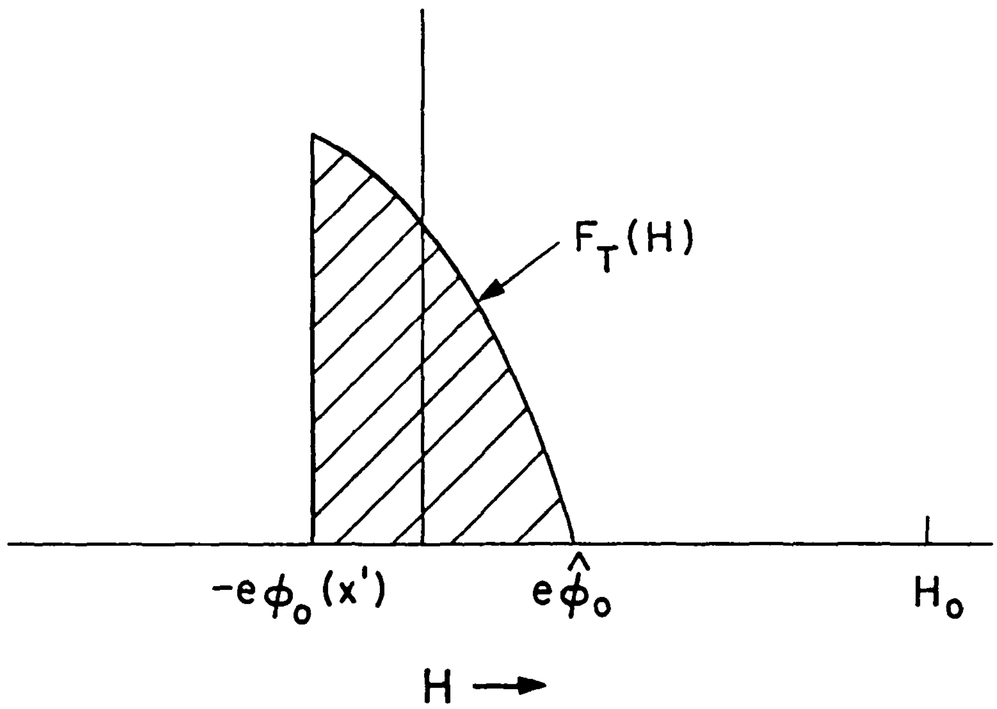

Fig. 2 Trapped electron distribution function in the wave frame [Eq. (37)].

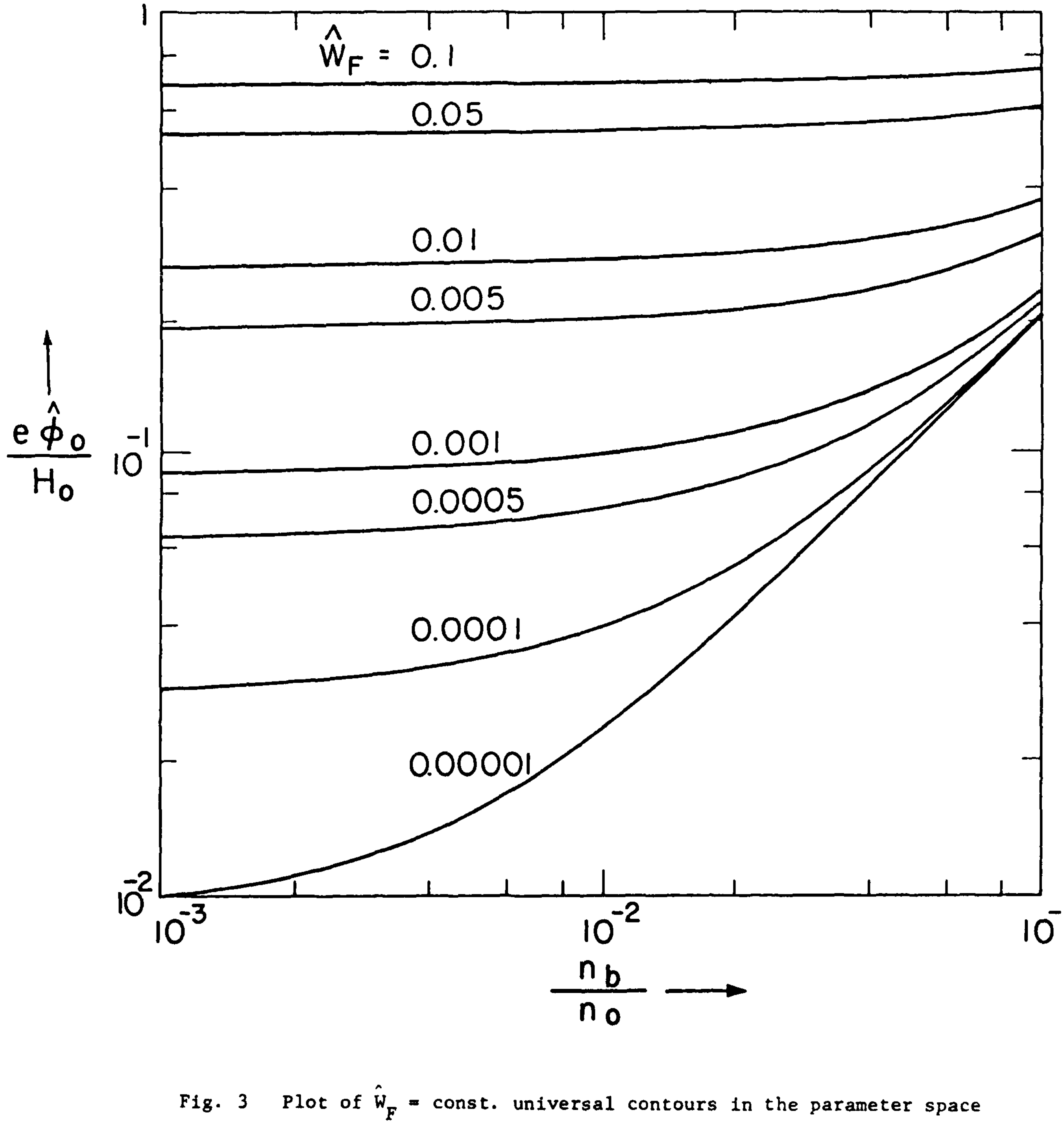

Fig. 3 Plot of $\hat{W}_F$ = const. universal contours in the parameter space ($e\hat{\phi}_0/H_0$, n_b/n_0) [Eq. (42)].

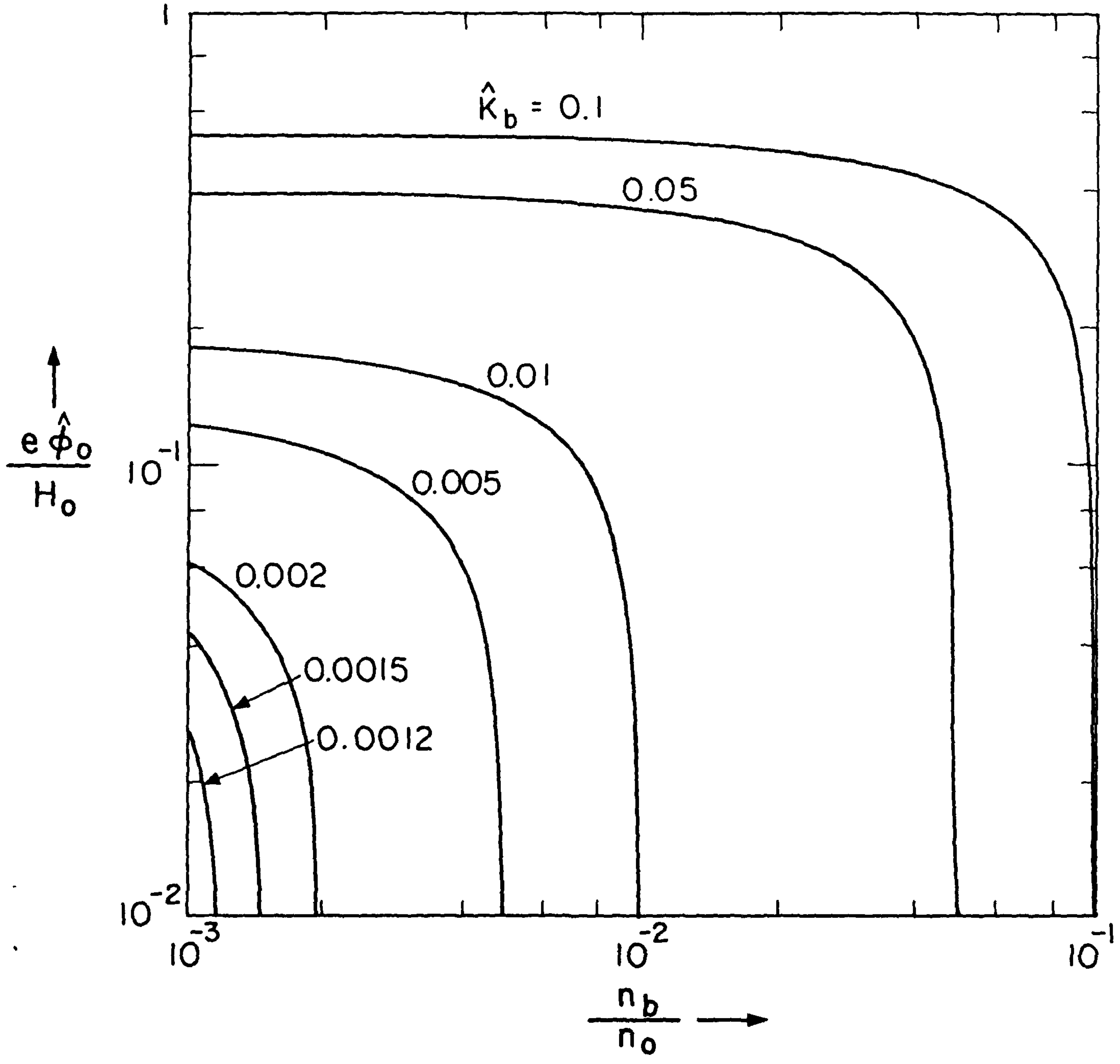

Fig. 4 Plot of $\hat{K}_b$ = const. universal contours in the parameter space $(e\hat{\phi}_0/H_0,\ n_b/n_0)$ [Eq. (53)].

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